A THEOREM ON ELECTRICAL NETWORKS WITH AN APPLICATION TO FILTERS

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Summary

A new general theorem on electrical networks is given (printed in italics immediately below).

The proof of this theorem is given with the aid of the recent development of the symbolic calculus.

The theorem is applied to an ordinary $L\,C\,r$ circuit.

A further application of the theorem is given to filters. If the „indicial admittance" of the low pass filter is given, the theorem enables one at once to write down the „indicial admittance" of a high pass filter in a new form, being an integral containing two Besselfunctions.

The theorem is *mutatis mutandis* also applicable to mechanical systems.

It is the purpose of the present note to prove the following general *Theorem:*

Be given an arbitrary passive, linear and constant network, which is originally at rest. Let a unit E.M.F. (zero before, unity after the time $t = 0$*) be applied in the m'th branch. Let the current in the k'th branch, resulting from this E.M.F., be* $i_k = i_k(t)$*. Further consider a second network derived from the first in the following way: every L is replaced by a C*, every C by a L* in such a way that*

$$\frac{1}{\omega_0 C^*} = \omega_0 L,$$

$$\frac{1}{\omega_0 L^*} = \omega_0 C,$$

(where ω_0 is an arbitrary frequency), while all the resistances are left unaltered:

$$r^* = r.$$

The second system is therefore derived from the first one by a process of „reflection" with regard to the frequency ω_0.

Let again in this reflected system a unit E.M.F. be applied in the m'th branch. Let the current in the k'th branch be $i_k^*(t)$. Then the theorem states that the current $i_k^*(t)$ in the reflected system can be obtained from the current $i_k(t)$ in the original circuit by the following integral:

$$i_k^*(t) = \int_{\tau=0}^{\tau=\infty} \omega_0 \sqrt{\frac{t}{\tau}} \cdot J_1(2\omega_0\sqrt{t\tau}) \cdot i_k(\tau)\, d\tau, \qquad \text{(I)}$$

where J_1 is the Besselfunction of order unity.

If, moreover, initially $i_k(t)$ contains no impuls, the theorem simplifies to:

$$i_k^*(t) = \int_{\tau=C}^{\tau=\infty} J_0(2\omega_0\sqrt{t\tau}) \cdot i_k'(\tau)\, d\tau \qquad \text{(II)}$$

where

$$i_k'(\tau) = \frac{d\, i_k(\tau)}{d\tau}.$$

The proof of this theorem can be given in the following way.

Let, with the usual notations, the first system be represented by the following set of linear simultaneous differential equations with constant coefficients:

$$\left.\begin{aligned}
&\left(L_{11}D + r_{11} + \frac{1}{C_{11}D}\right)i_1 + \ldots. + \left(L_{1n}D + r_{1n} + \frac{1}{C_{1n}D}\right)i_n = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0,\\
&\left(L_{m1}D + r_{m1} + \frac{1}{C_{m1}D}\right)i_1 + \ldots. + \left(L_{mn}D + r_{mn} + \frac{1}{C_{mn}D}\right)i_n = [1],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0,\\
&\left(L_{n1}D + r_{n1} + \frac{1}{C_{n1}D}\right)i_1 + \ldots. + \left(L_{nn}D + r_{nn} + \frac{1}{C_{nn}D}\right)i_n = 0,
\end{aligned}\right\} \qquad \text{(1)}$$

where D means d/dt and [1] represents the unit E.M.F. in the m'th branch, i.e. an E.M.F., which at the time $t = 0$ suddenly jumps from zero to unity.

In order to treat our problem operationally we multiply all equations of (1) with e^{-pt} and integrate between $t = 0$ and $t = \infty$.

Calling

$$\int_0^\infty e^{-pt}\, i_k(t)\, dt = X_k(p),$$

and noting that

$$\int_0^\infty e^{-pt}\, [1]\, dt = \frac{1}{p},$$

we obtain:

$$\left.\begin{aligned}
&\left(p\,L_{11} + r_{11} + \frac{1}{p\,C_{11}}\right)X_1(p) + \ldots + \left(p\,L_{1n} + r_{1n} + \frac{1}{p\,C_{1n}}\right)X_n(p) = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 0\\
&\left(p\,L_{m1} + r_{m1} + \frac{1}{p\,C_{m1}}\right)X_1(p) + \ldots + \left(p\,L_{mn} + r_{mn} + \frac{1}{p\,C_{mn}}\right)X_n(p) = \frac{1}{p}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 0\\
&\left(p\,L_{n1} + r_{n1} + \frac{1}{p\,C_{n1}}\right)X_1(p) + \ldots + \left(p\,L_{nn} + r_{nn} + \frac{1}{p\,C_{nn}}\right)X_n(p) = 0.
\end{aligned}\right\} \quad (2)$$

Hence $X_k(p)$ is found to be

$$X_k(p) = \frac{1}{p}\,\frac{\Delta'_{km}(p)}{\Delta_{nn}(p)}$$

where Δ_{nn} is the general determinant of the system, while Δ'_{km} is the minor belonging to the k'th column and m'th row.

Hence

$$p\,X_k(p) = \frac{\Delta'_{km}(p)}{\Delta_{nn}(p)},$$

or

$$p\,X_k(p) = p\int_0^\infty e^{-pt}\, i_k(t)\, dt = \frac{\Delta'_{km}(p)}{\Delta_{nn}(p)}, \quad (3)$$

which we write symbolically

$$i_k(t) \doteqdot \frac{\Delta'_{km}(p)}{\Delta_{nn}(p)}. \text{ [1]} \tag{3a}$$

Now consider a second (reflected) network, which we will denote with an asterisk * and which is derived from the first one by replacing every L by a C^* and every C by a L^* in such a way that

$$\left.\begin{aligned} \omega_0 L &= \frac{1}{\omega_0 C^*} \\ \omega_0 C &= \frac{1}{\omega_0 L^*} \\ r &= r^* \end{aligned}\right\} \tag{4}$$

where ω_0 is an arbitrary chosen frequency.

We can say that the second (reflected) network is obtained from the original one by reflecting the latter with respect to the frequency ω_0. Let again a unit E.M.F. be applied in the m'th branch of this reflected network. Then, writing the first equation only, we have for the reflected network, instead of (1):

$$\left.\begin{aligned} &\left(\frac{1}{C^*_{11} D} + r^*_{11} + L^*_{11} D\right) i^*_1 + \ldots + \left(\frac{1}{C^*_{1n} D} + r^*_{1n} + L^*_{1n} D\right) i^*_n = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{1*}$$

Again multiply all equations of (1*) throughout with e^{-pt} and integrate between $t = 0$ and $t = \infty$.

Then calling

$$\int_0^\infty e^{-pt} i^*_k(t)\, dt = X^*_k(p),$$

we obtain instead of (2) (writing again the first equation only)

$$\left.\begin{aligned} &\left(\frac{1}{p C^*_{11}} + r^*_{11} + p L^*_{11}\right) X^*_1(p) + \ldots + \left(\frac{1}{p C^*_{1n}} + r^*_{1n} + p L^*_{1n}\right) X^*_n(p) = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{2*}$$

1) We use here the notation introduced in Balth. van der Pol, On the operational solution of linear differential equations, Phil. Mag. **8**, 861, 1929.

or, remembering (4)

$$\left.\begin{aligned}\left(\frac{\omega_0^2}{p}L_{11}+r_{11}+\frac{p}{\omega_0^2 C_{11}}\right)X_1^*(p)+\ldots.+\left(\frac{\omega_0^2}{p}L_{1n}+r_{1n}+\frac{p}{\omega_0^2 C_{1n}}\right)X_n^*(p)=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}\quad(2a^*)$$

The left hand members of (2a*) are the same as in (2), only p has been replaced by ω_0^2/p, whereas the right hand members remain unchanged.

We thus obtain, instead of (3)

$$p\,X_k^*(p)=p\int_0^\infty e^{-pt}\,i_k^*(t)\,dt=\frac{\Delta'_{km}\left(\frac{\omega_0^2}{p}\right)}{\Delta_{nn}\left(\frac{\omega_0^2}{p}\right)}\,,\qquad(3^*)$$

or:

$$i_k^*(t)\doteqdot\frac{\Delta'_{km}\left(\frac{\omega_0^2}{p}\right)}{\Delta_{nn}\left(\frac{\omega_0^2}{p}\right)}\,.$$

Hence the symbolic representation of $i_k^*(t)$ is obtained from the symbolic representation of $i_k(t)$ by simply replacing in the latter p by ω_0^2/p. As the symbolic representation of $i_k(t)$ is given by

$$i_k(t)\doteqdot p\int_0^\infty e^{-pt}\,i_k(\tau)\,d\tau,$$

we can at once write down the symbolic representation for $i_k^*(t)$ in the form

$$i_k^*(t)\doteqdot\frac{\omega_0^2}{p}\int_0^\infty e^{-\frac{\omega_0^2\tau}{p}}\,i_k(\tau)\,d\tau=\int_0^\infty\frac{\omega_0^2\tau}{p}\,e^{-\frac{\omega_0^2\tau}{p}}\cdot\frac{i_k(\tau)}{\tau}\,d\tau.\qquad(5)$$

Now the original function of t belonging to the „image" $1/p\,e^{-1/p}$ [1]) is known to be

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1) l. c. page 878.

$$\frac{1}{p} e^{-\frac{1}{p}} \doteq \sqrt{t} \,.\, J_1 (2\sqrt{t})$$

and therefore

$$\frac{\omega_0^2 \tau}{p} e^{-\frac{\omega_0^2 \tau}{p}} \doteq \sqrt{\omega_0^2 t \tau} \,.\, J_1 (2\sqrt{\omega_0^2 t \tau}).$$

Hence, interpreting (5) backwards, we have

$$i_k^* (t) = \int_{\tau=0}^{\tau=\infty} \omega_0 \sqrt{\frac{t}{\tau}} \cdot J_1 (2\omega_0 \sqrt{t\tau}) \,.\, i_k (\tau)\, d\tau, \qquad \text{(I)}$$

which is the first part of the theorem stated.

When $i_k(\tau)$ contains no impuls, we can further integrate (I) by parts as follows (writing $\tau = s^2$):

$$i_k^* (t) = \int_{s=0}^{s=\infty} 2\omega_0 \sqrt{t} \,.\, J_1 (2\omega_0 \sqrt{t} \,.\, s) \,.\, i_k (s^2)\, ds =$$

$$= - \int_{s=0}^{s=\infty} i_k (s^2) \,.\, \frac{d}{ds} J_0 (2\omega_0 \sqrt{t}\, s) ds =$$

$$= - i_k (s^2) \,.\, J_0 (2\omega_0 \sqrt{t}\, s) \Big|_{s=0}^{s=\infty} + \int_{s=0}^{s=\infty} J_0 (2\omega_0 \sqrt{t}\, s) \,.\, \frac{d}{ds} i_k (s^2) \,.\, ds$$

and, as the system was originally considered to be at rest, the integrated part drops out and we are left with

$$i_k^* (t) = \int_{\tau=0}^{\tau=\infty} J_0 (2\omega_0 \sqrt{t\tau}) \,.\, i_k' (\tau)\, d\tau \qquad \text{(II)}$$

where

$$i_k' (\tau) = \frac{d\, i_k(\tau)}{d\tau},$$

which is the second part of the theorem stated.

It will be obvious that, if the original circuit is reflected *twice* with respect to the same frequency ω_0 the original circuit will again result. This is confirmed by the theorem (I), because the twofold application of the theorem (I) yields, after a slight transformation, the wellknown H a n k e l-identity [1])

$$\int_{\tau=0}^{\tau=\infty} \int_{\tau'=0}^{\tau'=\infty} f(\tau) \, J_n(2\sqrt{\tau\tau'}) . J_n(2\sqrt{t\tau'}) \, d\tau \, d\tau' = f(t).$$

Having thus proved the theorem which, mutatis mutandis, applies also to mechanical systems, let us consider two electrical examples. First consider a simple series circuit of a L and C to which a unit E.M.F. is applied. The current $i(t)$ in this circuit is given symbolically by

$$i(t) \doteqdot p\,X(p) = \frac{1}{pL + \frac{1}{Cp}} = \frac{1}{\omega_1 L} \cdot \frac{\omega_1 p}{p^2 + \omega_1^2}, \tag{6}$$

where ω_1^2 is written for $(CL)^{-1}$.

The original of (6) is well-known and is

$$i(t) = \frac{1}{\omega_1 L} \sin \omega_1 t. \tag{7}$$

The reflected circuit is obtained from the original one by replacing the C by a L^* and the L by a C^* such that

$$\omega_0 L = \frac{1}{\omega_0 C^*},$$

$$\omega_0 C = \frac{1}{\omega_0 L^*}.$$

Hence the current in the reflected circuit becomes symbolically from (6)

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1) G N. W a t s o n, *Bessel Functions*, page 453, (Cambridge, 1922).

$$i^*(t) \div p\,X^*(p) = \frac{1}{\frac{1}{pC^*} + pL^*} = \frac{1}{\omega_1 L}\,\frac{\omega_1 \frac{\omega_0^2}{p}}{\frac{\omega_0^4}{p^2} + \omega_1^2}\,, \qquad (6^*)$$

which is (6), but p replaced by ω_0^2/p.

If, moreover, we reflect the circuit not with respect to an arbitrary frequency ω_0, but with respect to its own natural frequency ω_1, we have $\omega_0 = \omega_1$, and the reflected system is equal to the original one and we must therefore have

$$i(t) = i^*(t)$$

and hence (6*) identical with (6).

As (7) contains no impuls, we may apply the second part of our theorem (II); writing $\omega_1 = \omega_0 = \omega$, we must therefore have the identity:

$$\frac{1}{\omega L}\,sin\,\omega t = \int_{\tau=0}^{\tau=\infty} J_0\,(2\,\omega\,\sqrt{t\tau}).\,\frac{1}{\omega L}\,\frac{d}{d\tau}\,sin\,(\omega\tau).\,d\tau$$

or, with a slightly different notation, the following identity must exist:

$$sin\,\alpha = \int_{s=0}^{s=\infty} J_0\,(2\,\sqrt{\alpha\,s}).\,cos\,s\,ds. \qquad (8)$$

In fact (8) constitutes a special case of the more general relation

$$\int_0^\infty J_{2n}\,(2\,\sqrt{\alpha\,s}).\,J_n\,(s)\,ds = J_n\,(\alpha)$$

found before [1]) and our theorem is thus verified in a special and simple case.

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1) Balth. van der Pol and K. F. Niessen, Phil. Mag. **13**, 558, 1932.

2) J. R. Carson and O. J. Zobel, Bell Syst. Techn. Journal, July 1923 page 7 or the book of J. R. Carson, *Electric Circuit Theory*, page 122 (New York, 1926).

Finally consider as second example a low pass filter (fig. 1), which

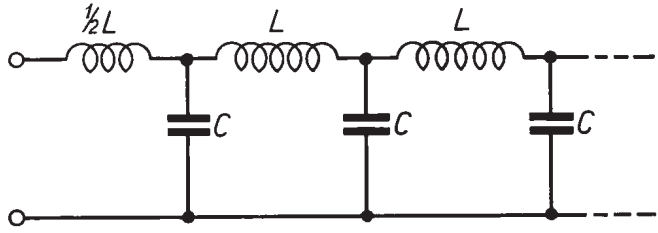

Fig. 1. Lowpass filter

we will reflect with respect to its cut-off frequency, so that it becomes a high pass filter. The current in the n'th section under unit E.M.F. applied at the beginning of the filter is numerically equal to the „indicial admittance" and is known to be:

$$i_n(t) = \sqrt{\frac{C}{L}} \int_0^{x=\omega_1 t} J_{2n}(x)\, dx \quad [2]) \tag{9}$$

where $\omega_1 = \frac{1}{2\sqrt{LC}}$ equals the cut-off angular frequency.

We will now, with the aid of our theorem, derive from this indicial admittance of a low pass filter a new expression for the indicial admittance of a high pass filter.

The high pass filter (fig. 2) with the same cut-off frequency ω_1 is obtained from the low pass filter by reflecting it with respect to $\omega_0 = \omega_1$. As the indicial admittance of a low pass filter, as given by (9), contains no impuls ,we can again apply the second part of the theorem (II). We have from (9)

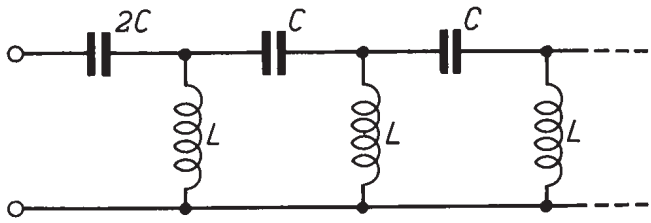

Fig. 2. Highpass filter

$$\frac{di_n(t)}{dt} = \sqrt{\frac{C}{L}}\,\omega_1\,J_{2n}(\omega_1 t)\,.$$

Hence the indicial admittance $i_n^*(t)$ of the high pass filter with cut-off frequency ω_1 is, with our theorem, at once found in the new form

$$i_n^*(t) = \int_{\tau=0}^{\tau=\infty} J_0(2\omega_1\sqrt{t\tau})\,.\,\sqrt{\frac{C}{L}}\,\omega_1\,J_{2n}(\omega_1\tau)\,d\tau$$

or, writing $\omega_1\tau = s$,

$$i_n^*(t) = \sqrt{\frac{C}{L}}\int_{s=0}^{s=\infty} J_0(2\sqrt{\omega_1 ts})\,.\,J_{2n}(s)\,.\,ds$$

Eindhoven, 4th April 1934.